JOURNAL OF APPROXIMATION THEORY 6, 312–315 (1972)

On a Problem of C. Renyi Concerning Julia Lines*†

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Communicated by Oved Shisha

Received September 20, 1970

DEDICATED TO PROFESSOR J. L. WALSH ON THE OCCASION OF HIS 75TH BIRTHDAY

1. INTRODUCTION

In W. K. Hayman's function theory problem book [1] the following problem (attributed to C. Renyi) is stated [1, p. 10, prob. 2.4]:

> Can an entire function $E(z)$ have one finite exceptional value for one Julia line (for the definition of a Julia line see [1, p. 10]) and another finite exceptional value for some other Julia line?

The following theorem answers the above question affirmatively.

THEOREM 1. *There exists an entire function $E(z)$ which has the positive real axis as a Julia line with the exceptional value one and the negative real axis as a Julia line with the exceptional value zero.*

The proof makes use of some elementary techniques in conformal mapping in conjunction with a little known, but very powerful approximation theorem of Keldyŝ and Mergelyan [2, p. 335, Theorem 1.3].

* This research was partially supported by the National Science Foundation, Grants GP 8787 and GP 20703.

† Presented to the American Mathematical Society, October 26, 1968.

2. Proof of Theorem 1

Before going into the proof we shall make a few definitions and then say a few words concerning the idea underlying the proof.

Let

$$D = \{z: |z| < 1\} \cup \{z: |\arg z| < \pi/4\} \cup \{z: |\arg z - \pi| < \pi/4\}$$

and let

$$\tilde{D} = \{w = u + iv: |u| < \sigma(v), -\infty < v < \infty\},$$

where $\sigma(v)$ is a function (to be specified later) which has the properties:

(i) $\sigma(v)$ is even,

(ii) $\sigma(v) > 0$, and

(iii) $\sigma(v) \uparrow +\infty$ as $|v| \uparrow +\infty$.

Let $c(z)$ conformally map D onto $\tilde{D}$ with $c(0) = 0$, $c(+\infty) = i\infty$ and $c(-\infty) = -i\infty$. Such a map exists since there exists a conformal map $h(z)$ of the part of D in the upper half plane onto the part of $\tilde{D}$ in the right half plane, sending 0 to 0, $+\infty$ to $i\infty$, and $-\infty$ to $-i\infty$. The mapping $c(z)$ is obtained by extending $h(z)$ by the reflection principle. Let

$$\tilde{G} = \{w: |w - 1/2| < 1/2\},$$

and let $g(z)$ conformally map D onto $\tilde{G}$ with $g(0) = 1/2$, $g(\infty) = 1$ and $g(-\infty) = 0$.

The idea of the proof is to first consider the function $f(z) = [e^{c(z)} + g(z)]$ and show that it has the positive real axis as a Julia line with exceptional value one and the negative real axis as a Julia line with exceptional value zero. Then, applying a theorem of Keldyŝ and Mergelyan, we can find an entire function $E(z)$ which, when restricted to D, approximates $f(z)$ so well that $E(z)$ also has the positive real axis as a Julia line with exceptional value one and the negative real axis as a Julia line with exceptional value zero.

To see that $f(z)$ has the positive real axis as a Julia line with exceptional value one, we must show that for all sufficiently small positive α, the image under $f(z)$ of $S_\alpha (= \{z: |\arg z| < \alpha\})$ is the w-plane punctured at one. Let $\tau(r)$ be the radius of the smallest disk in the w-plane, about $w = 1$, which contains $g[S_\alpha \cap \{|z| > r\}]$. Clearly $\tau(r) \to 0$ as $r \to \infty$. We note that if we choose $\sigma(|v|)$ so as to grow sufficiently slowly, we can make sure that, for sufficiently large r, the image of $(S_\alpha \cap \{|z| < 2r\})$ under $e^{c(z)}$ will be contained in $\{w: 2\tau(r) < |w| < 2/\tau(r)\}$. Therefore, for sufficiently large r, the image of $(S_\alpha \cap \{r < |z| < 2r\})$ under $f(z) = [e^{c(z)} + g(z)]$ is contained in

$$\{w: \tau(r) < |w - 1| < 2/\tau(r) + \tau(r)\};$$

this follows, since for large values of r, the effect of adding $g(z)$ is essentially to translate the image of $S_\alpha \cap \{r < |z| < 2r\}$ under $e^{c(z)}$ one unit to the right. Let $\kappa(n)$ be the largest positive number κ such that the set

$$\{w: |u| < \kappa, 2\pi(n-1) < v < 2\pi n\}$$

is contained in $c(S_\alpha)$. Since $\sigma(v) \to \infty$ as $|v| \to \infty$, $\kappa(n) \to \infty$ as $n \to \infty$ (this follows rather easily by the standard techniques for investigating the boundary behavior of conformal maps—however, these techniques are not generally well-known, and the reader who is not familiar with them is referred to Lemma 1 in Section 3 and the subsequent remarks). Since $\kappa(n) \to \infty$ as $n \to \infty$, it is clear that the image of S_α under $f(z)$ covers $\{0 < |w-1| < \infty\}$ and, from previous remarks, does not cover $w = 1$. By an argument completely analogous to the above, one can also show that $f(z)$ has the negative real axis as a Julia line with exceptional value zero.

Now, by an approximation theorem of Keldyŝ and Mergelyan [2, p. 335, Theorem 1.3] it follows that one can choose $\sigma(v)$ in such a way that the corresponding $f(z)$ can be approximated by an entire function $E(z)$ having the properties stated in Theorem 1.

Remarks. Using a generalization of the above argument, one can construct an entire function having *any* finite set (indeed, even certain countable sets) of rays as Julia lines with *any* prescribed exceptional value for each Julia line. In addition, our method can also easily be adapted to answer affirmatively the analogous question for Julia curves.

3. Statement and Proof of Lemma 1

LEMMA 1. *Let D be a domain bounded by J_1 (the nonnegative real axis) and J_2 (the curve $y = h(x)$, where h is a real, continuous function on $[0, \infty)$ with $h(0) = 0$, $\lim_{x\to\infty} h(x) = +\infty$). Let $\omega(z)$ be the harmonic measure of J_1 (i.e., the harmonic function which solves the Dirichlet problem in D for the values one on J_1 and zero on J_2). Then $\{z: \omega(z) = \lambda\}$, for $0 < \lambda < 1$, is a curve in D whose y-coordinate tends to $+\infty$ as its x-coordinate tends to $+\infty$.*

Proof. Let $f(z)$ be the conformal map from the upper half plane (UHP) onto D such that $f(0) = \infty$, $f(\infty) = 0$, $f(R^+) = J_2$ and $f(R^-) = J_1$ (where $R^+ = \{z: y = 0, x \geqslant 0\}$ and $R^- = \{z: y = 0, x \leqslant 0\}$).

The conclusion of the lemma is equivalent to the statement: $\operatorname{Im} f(z) \to +\infty$ as z approches the origin along $\arg z = \lambda\pi$ (since harmonic measure is a conformal invariant, and f maps $\arg z = \lambda\pi$ onto $\{z: \omega(z) = \lambda\}$).

ASSERTION 1. *Given any real M and any $\epsilon > 0$, one can find an r such that* $\operatorname{Im} f(z) > M$ *in* $N_\epsilon(r) = \{z: |z| < r, \epsilon < \arg z < \pi - \epsilon\}$.

Proof of Assertion 1. From the way $f(z)$ was defined, it is clear that $\operatorname{Im} f(z) \to +\infty$ as $z \to 0$ along R^+, and that there exists an $\tilde{r} > 0$ such that $\operatorname{Im} f(z) > 2M\pi/\epsilon$ on $(0, \tilde{r})$. Let $\tilde{\omega}(z)$ be the harmonic measure of $(0, \tilde{r})$ with respect to the UHP. By the maximum principle

$$(\operatorname{Im} f(z) - (2M\pi/\epsilon)\,\tilde{\omega}(z)) \geqslant 0$$

in UHP. Since $h(z) = \tilde{\omega}(z) - (1 - (\arg z)/\pi)$ has boundary values $\equiv 0$ in some neighborhood of zero, $h(z)$ is continuous to the boundary in some neighborhood of zero and, hence, $h(z)$ tends to zero uniformly as $z \to 0$. This means, however, that there exists an r such that $(2\pi M/\epsilon)\,\tilde{\omega}(z) \geqslant M$ for $z \in N_\epsilon(r)$ since, in $\{z: \epsilon < \arg z < \pi - \epsilon\}$, the function $(2M/\epsilon)(\pi - \arg z)$ is greater than $2M$. The assertion now follows since we proved above that $\operatorname{Im} f(z)$ dominates $(2\pi M/\epsilon)\,\tilde{\omega}(z)$ in the UHP.

Since M and ϵ were arbitrary in Assertion 1, it follows that $\operatorname{Im} f(z)$ must tend to $+\infty$ as $z \to 0$ along $\arg z = \lambda\pi$.

Remark. To see that $\kappa(n) \to \infty$ as $n \to \infty$, one combines Lemma 1 with the fact that the level curves of the harmonic measure of the upper boundary of $\tilde{D}$ are asymptotic to the half rays $\arg z = \theta$.

REFERENCES

1. W. K. HAYMAN, "Research Problems in Function Theory," Athlone Press, London, 1967.
2. S. N. MERGELYAN, Uniform approximations to functions of a complex variable (in Russian), *Uspehi Mat. Nauk (N. S.)* 7 (1952), 31–122; *Amer. Math. Soc. Transl.* **3** (1962), 294–391.